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On some nonlinear extensions of the angular momentum algebra

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Abstract. Deformations of the Lie algebras so(4), so(3,1) and e(3) that leave their so(3) subalgebra undeformed and preserve their coset structure are considered. It is shown that such deformed algebras are associative for any choice of the deformation parameters. Their Casimir operators are obtained and some of their unitary irreducible representations are constructed. For vanishing deformation, the unitary irreducible representations of the deformed algebras transform into those of the corresponding Lie algebras that contain each of the so(3) unitary irreducible representations at most once. It is also proved that similar deformations of the Lie algebras su(3), sl(3,ℝ) and of the semidirect sum of an Abelian algebra t(5) and so(3) do not lead to associative algebras.

1. Introduction

In recent years many works have been devoted to the study of deformations and extensions of Lie algebras and their applications in various branches of physics. Some of the studies have been are carried out in the mathematically well-defined framework of quasi-triangular Hopf algebras and deal with the so-called quantum groups and q-algebras (Drinfeld 1986, Jimbo 1985); other studies put less emphasis on the coalgebra structure, which is often dropped completely, but instead insist on preserving some other property of the Lie algebra that is deformed. In this second category one finds, for instance, some deformed algebras that can be realized in terms of deformed creation and annihilation operators (Fairlie and Zachos 1991, Fairlie and Nuyts 1994).

In the same class there are also deformed algebras that have a coset structure $g_{\mathrm{d}} = h + v_{\mathrm{d}}$ and which can be viewed as nonlinear extensions of an ordinary Lie algebra h (Roček 1991). This means that their generators can be separated into the generators E_i of h and some operators E_α transforming as a representation of h, and commuting among themselves to give a function of the E_i's only. In other words, they satisfy the commutation relations

$$[E_i, E_j] = c_{ij}^k E_k \qquad [E_i, E_\alpha] = (\tau_i)_\alpha^\beta E_\beta \qquad [E_\alpha, E_\beta] = f_{\alpha\beta}(E_i) \qquad (1.1)$$

where c_{ij}^k and $(\tau_i)_\alpha^\beta$ are the structure constants and matrix representations of h respectively, while $f_{\alpha\beta}(E_i)$ are formal power series in the E_i's that are constrained by the associativity requirement, i.e. Jacobi identities, and by the condition that for some limiting values of the parameters, g_{d} transforms into some Lie algebra g with coset structure $g = h + v$. The

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simplest example corresponding to $h = \mathrm{u}(1)$ and $g = \mathrm{su}(2)$ or su(1,1) has been discussed in detail (Polychronakos 1990, Roček 1991).

An interest in such constructions in the infinite-dimensional case was shown some years ago in the context of quantum field theory and statistical physics models where they are known as W-algebras (Zamolodchikov 1986, Schoutens *et al* 1989). More recently, finite versions of these W-algebras have been introduced by considering symplectic reductions of finite-dimensional simple Lie algebras (Tjin 1992, de Boer and Tjin 1993); in particular, it was shown that the finite $W_3^{(2)}$-algebra, known as $\overline{W}_3^{(2)}$, is related to the simplest example of the deformed algebra g_{d} mentioned above.

The $\overline{W}_3^{(2)}$-algebra has recently appeared in various physical problems. Let us mention three of them. First, the deformed su(2) (or su(1,1)) algebra may be considered as a dynamical symmetry algebra in some quantum many-body models with symmetry-preserving Hamiltonians, such as those occurring in quantum optics (Karassiov 1994, Karassiov and Klimov 1994 and references quoted therein); second, it is related to generalized deformed parafermions and, through the introduction of a Fermi- like oscillator Hamiltonian, provides a new algebraic description of the bound-state spectra of the Morse and Pőschl–Teller potentials (Quesne 1994). Moreover, by superposing these generalized deformed parafermions with ordinary bosons, one obtains some deformations of parasupersymmetric quantum mechanics with new and non-trivial properties (Beckers *et al* 1995). Further, the algebra $\overline{\mathrm{W}}_3^{(2)}$ may be considered as the symmetry algebra of the two-dimensional anisotropic harmonic oscillator with frequency ration 2:1 (Bonatsos *et al* 1994).

Motivated by these applications, in this paper we shall consider another class of examples of deformed algebras g_{d} that should be physically relevant. The class corresponds to the case where the undeformed subalgebra h is the angular momentum algebra so(3) and the deformed subspace v_{d} is spanned by the $2\lambda + 1$ components of an so(3) irreducible tensor of rank λ. Special emphasis will be laid on the vector ($\lambda = 1$) and quadrupole ($\lambda = 2$) cases, corresponding to deformations of so(4) and su(3) (or of their non-compact or non-semisimple variants) respectively.

In the following section, the relevant associativity conditions are established. The vector and quadrupole cases are then studied in detail in sections 3 and 4 respectively. Finally, section 5 contains the conclusion.

2. Nonlinear extensions of so(3)

Let the generators E_i and E_α, introduced in the previous section, be the spherical components $L_m = (-1)^m L_{-m}^\dagger$, $m = +1, 0, -1$, and $T_\mu^\lambda = (-1)^\mu T_{-\mu}^{\lambda\dagger}$, $\mu = \lambda, \lambda-1, \ldots, -\lambda$, of an angular momentum operator and of an irreducible tensor of integer rank λ respectively. In such a case it is advantageous to write equation (1.1) in a coupled commutator form:

$$[L, L]_m^1 = -\sqrt{2} L_m \qquad (2.1a)$$

$$\left[L, T^\lambda\right]_M^\Lambda = -\sqrt{\lambda(\lambda+1)}\, \delta_{\Lambda,\lambda}\, \delta_{M,\mu}\, T_\mu^\lambda \qquad (2.1b)$$

$$\left[T^\lambda, T^\lambda\right]_M^\Lambda = f_M^\Lambda(L) \qquad (2.1c)$$

where $f^\Lambda(L)$ is an irreducible tensor of rank Λ, whose components can be written as formal power series in the vector operator L. The definition of coupled commutators and some of their properties are reviewed in appendix 1. From equation (A1.2) it can be seen that the values of Λ in equation (2.1c) are restricted to odd integers, $\Lambda = 1, 3, \ldots, 2\lambda - 1$.

It has been shown by Gaskell *et al* (1978) that the number of linearly independent irreducible tensors of rank Λ whose components are monomials of degree n in L is equal to one if $n = \Lambda + 2k$, $k = 0, 1, 2, \ldots$, and zero otherwise. Hence, the explicit form of the functions $f_M^\Lambda(L)$ is given by

$$f_M^\Lambda(L) = \gamma_\Lambda g_\Lambda(L^2) \left[\cdots \left[[L \times L]^2 \times L\right]^3 \times \cdots\right]_M^\Lambda \qquad \Lambda = 1, 3, \ldots, 2\lambda - 1 \tag{2.2}$$

where γ_1 is some real normalization constant, $\gamma_\Lambda = 1$ for $\Lambda \neq 1$,

$$g_\Lambda(L^2) = \sum_{k=0}^{\infty} a_k^{(\Lambda)} L^{2k} \qquad a_k^{(\Lambda)} \in \mathbb{R} \qquad a_0^{(1)} = +1, 0, \text{or} -1 \tag{2.3}$$

is a formal power series in the scalar operator $L^2 = \sum_m (-1)^m L_m L_{-m}$, and the last factor on the right-hand side of (2.2) is a 'stretched' product of Λ operators L. The deformed algebra g_d is therefore a $(2K + 1)$th-degree algebra where $K = 0, 1, 2, \ldots$, provided that $a_k^{(\Lambda)} = 0$ if $2k + \Lambda > 2K + 1$ and at least one $a_k^{(\Lambda)}$ with $2k + \Lambda = 2K + 1$ is different from zero. As $K = 0$ corresponds to an ordinary Lie algebra we shall henceforth refer to K as the deformation order.

We shall be concerned here with the cases where T^λ is an irreducible tensor of rank 1 (vector operator) or rank 2 (quadrupole operator). In the former case, we shall denote T_m^1 by A_m, $m = +1, 0, -1$; in equation (2.1c) Λ then takes the single value $\Lambda = 1$. In the latter case, we shall denote T_μ^2 by Q_μ, $\mu = +2, +1, 0, -1, -2$; in equation (2.1c), Λ can then take the values $\Lambda = 1$ and $\Lambda = 3$. For future reference it is helpful to state some relations among linearly dependent irreducible tensors:

$$[L \times L]_0^0 = -\frac{1}{\sqrt{3}} L^2 \tag{2.4a}$$

$$[L \times L]_m^1 = -\frac{1}{\sqrt{2}} L_m \tag{2.4b}$$

$$\left[[L \times L]^2 \times L\right]_m^1 = -\frac{2}{\sqrt{15}} L^2 L_m + \frac{1}{2}\sqrt{\frac{3}{5}} L_m \tag{2.4c}$$

$$\left[[L \times L]^2 \times L\right]_\mu^2 = -\sqrt{\frac{3}{2}} [L \times L]_\mu^2. \tag{2.4d}$$

The deformed algebra generated by L_m and T_μ^λ will be associative provided commutators (2.1) satisfy the Jacobi identity. For three irreducible tensors T^{λ_1}, U^{λ_2} and V^{λ_3}, of ranks λ_1, λ_2 and λ_3, respectively, the Jacobi identity can be written in a coupled form, as shown in equation (A1.5). If $\left(T^{\lambda_1}, U^{\lambda_2}, V^{\lambda_3}\right) = (L, L, L)$, $\left(L, L, T^\lambda\right)$ or $\left(L, T^\lambda, T^\lambda\right)$ equation (A1.5) is automatically satisfied, whereas if $\left(T^{\lambda_1}, U^{\lambda_2}, V^{\lambda_3}\right) = \left(T^\lambda, T^\lambda, T^\lambda\right)$ it leads to the set of conditions

$$\sum_{\Lambda_{12}=1,3}^{2\lambda-1} \left\{\delta_{\Lambda_{12},\Lambda_{23}} - 2(-1)^{\lambda-\Lambda} U(\lambda\lambda\Lambda\lambda; \Lambda_{12}\Lambda_{23})\right\} \left[T^\lambda, \left[T^\lambda, T^\lambda\right]^{\Lambda_{12}}\right]_M^\Lambda = 0$$
$$\Lambda_{23} = 1, 3, \ldots, 2\lambda - 1 \qquad \Lambda = |\lambda - \Lambda_{23}|, |\lambda - \Lambda_{23}| + 1, \ldots, \lambda + \Lambda_{23} \tag{2.5}$$

where $U(\lambda\lambda\Lambda\lambda; \Lambda_{12}\Lambda_{23})$ denotes a Racah coefficient in unitary form (Rose 1957). Note that for simplicity we shall now drop the component label of all irreducible tensors.

By using the numerical values of Racah coefficients one finds that the set of conditions (2.5) reduces to a single independent condition

$$\left[A, [A, A]^1\right]^0 = 0 \tag{2.6}$$

in the vector case, and to two independent conditions

$$2\sqrt{2}\left[Q, [Q, Q]^1\right]^1 + \sqrt{7}\left[Q, [Q, Q]^3\right]^1 = 0 \tag{2.7a}$$

$$\left[Q, [Q, Q]^1\right]^3 - 2\left[Q, [Q, Q]^3\right]^3 = 0 \tag{2.7b}$$

in the quadrupole case. In the next two sections we shall determine whether these identities are satisfied when the inner commutators are given by equations (2.1*c*), (2.2) and (2.3).

3. The vector case

In the vector case the deformed algebra g_d is defined by the commutation relations

$$[L, L]^1 = -\sqrt{2}L \tag{3.1a}$$

$$[L, A]^\Lambda = -\sqrt{2}\,\delta_{\Lambda,1}A \tag{3.1b}$$

$$[A, A]^1 = f^1(L) = -\sqrt{2}\,g(L^2)L = -\sqrt{2}\left(\sum_{k=0}^{\infty} a_k L^{2k}\right)L \tag{3.1c}$$

where the normalization constant γ_1 has been set equal to $-\sqrt{2}$. A Kth-order deformation corresponds to $a_K \neq 0$ and $a_{K+1} = a_{K+2} = \cdots = 0$.

3.1. Associativity condition

The algebra g_d is associative provided A satisfies equation (2.6). By using equations (3.1) and (A1.4), equation (2.6) can be rewritten as

$$\left[\left[A, g(L^2)\right]^1 \times L\right]^0 = 0. \tag{3.2}$$

In the undeformed case where $g(L^2) = a_0$ this condition is trivially fulfilled and g_d then reduces to an ordinary Lie algebra g. According to whether $a_0 = +1, -1$ or 0, g is the orthogonal algebra so(4), the pseudo-orthogonal algebra so(3,1) or the Euclidean algebra e(3), which is a semidirect sum of an Abelian algebra t(3) and so(3) (Biedenharn 1961, Naimark 1964, Böhm 1979).

For a first-order deformation, condition (3.2) reduces to

$$\left[\left[A, L^2\right]^1 \times L\right]^0 = 0. \tag{3.3}$$

From equations (2.4*a*), (A1.4), (3.1*b*) and (A1.1) one obtains

$$\left[A, L^2\right]^1 = 2\left\{A + \sqrt{2}\,[L \times A]^1\right\}. \tag{3.4}$$

Moreover, standard tensor algebra and recoupling techniques (Rose 1957) show that

$$[[L \times A]^1 \times L]^0 = [L \times [L \times A]^1]^0 = [[L \times L]^1 \times A]^0 = -\frac{1}{\sqrt{2}}[L \times A]^0 \tag{3.5}$$

where, in the last step, use has been made of equation (2.4*b*). By introducing equations (3.4) and (3.5) into the left-hand side of equation (3.3) one finds that the latter is identically satisfied.

It is now an easy task to show that if condition (3.2) is fulfilled for a Kth-order deformation then it is also satisfied for a $(K+1)$th-order deformation. From equation (A1.4) one indeed obtains

$$\begin{aligned}\left[[A, L^{2k+2}]^1 \times L\right]^0 &= \left[\left[L^{2k} \times [A, L^2]^1\right]^1 \times L\right]^0 + \left[\left[[A, L^{2k}]^1 \times L^2\right]^1 \times L\right]^0 \\ &= L^{2k}\left[[A, L^2]^1 \times L\right]^0 + \left[[A, L^{2k}]^1 \times L\right]^0 L^2.\end{aligned} \tag{3.6}$$

Hence, if the relation

$$\left[[A, L^{2k}]^1 \times L\right]^0 = 0 \tag{3.7}$$

is identically satisfied for $k = K$, then the same is true for $k = K + 1$. This completes the proof by induction of the following result.

Proposition 1. For any choice of the deformation parameters $a_k, k = 1, 2, \ldots$, equation (3.1) defines an associative algebra g_{d}, which is a deformed so(4), so(3,1) or e(3) algebra according to whether $a_0 = +1, -1$ or 0.

Remark 1. The first-order deformation of so(4) has already been encountered elsewhere. It is indeed the dynamical symmetry algebra of a particle moving in a three-dimensional space with constant curvature under the influence of a Coulomb potential (Higgs 1979, Leemon 1979, Granovskii *et al* 1992, de Vos and van Driel 1993). In such a case the deformation parameter a_1 is related to the space curvature.

3.2. *Casimir operators*

It is well known that the Lie algebras g = so(4), so(3,1) and e(3) have two independent Casimir operators, which may be written as

$$C_1 = a_0 L^2 + A^2 \qquad C_2 = L \cdot A \tag{3.8}$$

where, as usual, $L \cdot A$ denotes the scalar product $\sum_m (-1)^m L_m A_{-m}$. The purpose of this subsection is to show that the operators (3.8) can be deformed so as to provide Casimir operators of the deformed algebra g_{d}.

The case of the second Casimir operator is easily solved. One finds the following result.

Proposition 2. When transforming from g = so(4), so(3,1) or e(3) to g_{d}, defined in (3.1), the operator $L \cdot A$ remains a Casimir operator, which we shall denote by $C_{2\mathrm{d}}$.

Proof. By using (A1.4), (3.1) and (A1.2) one obtains

$$[A, L \cdot A]^1 = -\frac{1}{\sqrt{2}}[A, A]^1 + \frac{1}{\sqrt{2}} f(L^2)[L, L]^1 = 0. \tag{3.9}$$

This completes the proof as, by construction, $L \cdot A$ commutes with L. □

The case of the first Casimir operator is more involved and has not actually been solved in full generality. We conjecture that (i) for any choice of the real constants a_k in equation (3.1) it is possible to find some real constants b_k, $k = 1, 2, \ldots$, such that

$$C_{1\mathrm{d}} = h(L^2) + A^2 \qquad \text{where } h(L^2) = \sum_{k=1}^{\infty} b_k L^{2k} \tag{3.10}$$

is a Casimir operator of the deformed algebra g_{d}, and (ii) for a Kth-order deformation these constants are such that $b_k = 0$ for $k > K + 1$. We shall now proceed to prove that this conjecture is at least valid up to fourth order in the deformation.

For such a purpose we have first to determine the commutators of L^{2k} and A^2 with A. We state the results in the form of two lemmas.

Lemma 1. For any $k \in \mathbb{N}^+$ the generators of the deformed algebra g_{d}, defined in (3.1), satisfy the relation

$$\left[A, L^{2k}\right]^1 = \sum_{i=0}^{k-1} x_i^{(k)} L^{2i} A + \sum_{i=0}^{k-1} y_i^{(k)} L^{2i} [L \times A]^1 + \sum_{i=0}^{k-2} z_i^{(k)} L^{2i} \left[[L \times L]^2 \times A\right]^1 \tag{3.11}$$

where $x_i^{(k)}$, $y_i^{(k)}$, $i = 0, 1, \ldots, k-1$, and $z_i^{(k)}$, $i = 0, 1, \ldots, k-2$, are some real constants fulfilling the recursion relations

$$x_i^{(k)} = 2x_i^{(k-1)} + x_{i-1}^{(k-1)} + \tfrac{2}{3}\sqrt{2} y_{i-1}^{(k-1)} + 2\delta_{i,k-1} \qquad i = 0, 1, \ldots, k-1 \tag{3.12a}$$

$$y_i^{(k)} = 2\sqrt{2} x_i^{(k-1)} + y_i^{(k-1)} - \sqrt{\tfrac{3}{10}} z_i^{(k-1)} + y_{i-1}^{(k-1)} + 2\sqrt{\tfrac{2}{15}} z_{i-1}^{(k-1)} + 2\sqrt{2}\delta_{i,k-1}$$
$$i = 0, 1, \ldots, k-1 \tag{3.12b}$$

$$z_i^{(k)} = \sqrt{\tfrac{10}{3}} y_i^{(k-1)} - z_i^{(k-1)} + z_{i-1}^{(k-1)} \qquad i = 0, 1, \ldots, k-2 \tag{3.12c}$$

and the conditions $x_0^{(1)} = 2$, $y_0^{(1)} = 2\sqrt{2}$.

Lemma 2. The generators of the deformed algebra g_{d}, defined in (3.1), satisfy the relation

$$\left[A, A^2\right]^1 = -\sum_{i=0}^{\infty} u_i L^{2i} A - \sum_{i=0}^{\infty} v_i L^{2i} [L \times A]^1 - \sum_{i=0}^{\infty} w_i L^{2i} \left[[L \times L]^2 \times A\right]^1 \tag{3.13}$$

where u_i, v_i and w_i are some real constants defined by the formal series

$$u_i = \sum_{k=i}^{\infty} a_k \left(2x_i^{(k)} + \tfrac{1}{3}\sqrt{2} y_{i-1}^{(k)} + 2\delta_{k,i}\right) \tag{3.14a}$$

$$v_i = \sum_{k=i}^{\infty} a_k \left(\sqrt{2} x_i^{(k)} + \tfrac{3}{2} y_i^{(k)} - \tfrac{1}{2}\sqrt{\tfrac{3}{10}} z_i^{(k)} + \sqrt{\tfrac{2}{15}} z_{i-1}^{(k)} + 2\sqrt{2}\delta_{k,i}\right) \tag{3.14b}$$

$$w_i = \sum_{k=i+1}^{\infty} a_k \left(\sqrt{\tfrac{5}{6}} y_i^{(k)} + \tfrac{1}{2} z_i^{(k)}\right) \tag{3.14c}$$

Table 1. Solution of the recursion relations (3.12) up to $k = 5$.

	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$x_i^{(2)}$	4	$\frac{20}{3}$	—	—	—
$y_i^{(2)}$	$6\sqrt{2}$	$4\sqrt{2}$	—	—	—
$z_i^{(2)}$	$4\sqrt{\frac{5}{3}}$	—	—	—	—
$x_i^{(3)}$	8	$\frac{76}{3}$	14	—	—
$y_i^{(3)}$	$12\sqrt{2}$	$26\sqrt{2}$	$6\sqrt{2}$	—	—
$z_i^{(3)}$	$8\sqrt{\frac{5}{3}}$	$4\sqrt{15}$	—	—	—
$x_i^{(4)}$	16	$\frac{224}{3}$	88	24	—
$y_i^{(4)}$	$24\sqrt{2}$	$88\sqrt{2}$	$68\sqrt{2}$	$8\sqrt{2}$	—
$z_i^{(4)}$	$16\sqrt{\frac{5}{3}}$	$16\sqrt{15}$	$8\sqrt{15}$	—	—
$x_i^{(5)}$	32	$\frac{592}{3}$	368	$\frac{680}{3}$	$\frac{110}{3}$
$y_i^{(5)}$	$48\sqrt{2}$	$248\sqrt{2}$	$352\sqrt{2}$	$140\sqrt{2}$	$10\sqrt{2}$
$z_i^{(5)}$	$32\sqrt{\frac{5}{3}}$	$48\sqrt{15}$	$160\sqrt{\frac{5}{3}}$	$40\sqrt{\frac{5}{3}}$	—

in terms of the solution of equation (3.12).

The proofs of lemmas 1 and 2 are sketched in appendix 2 and the solutions obtained for $x_i^{(k)}$, $y_i^{(k)}$ and $z_i^{(k)}$, $k \leqslant 5$, by solving equation (3.12) are listed in table 1.

Finally, by combining lemmas 1 and 2 the following result can be easily derived (for details see appendix 2).

Proposition 3. Up to fourth order in the deformation, the operator $C_{1\mathrm{d}}$ defined in equation (3.10), where

$$\begin{aligned} &b_1 = a_0 + a_1 \qquad b_2 = \tfrac{1}{2}a_1 + \tfrac{4}{3}a_2 - \tfrac{1}{3}a_3 + \tfrac{8}{15}a_4 \qquad b_3 = \tfrac{1}{3}a_3 + \tfrac{5}{3}a_3 - \tfrac{16}{15}a_4 \\ &b_4 = \tfrac{1}{4}a_3 + 2a_4 \qquad b_5 = \tfrac{1}{5}a_4 \qquad b_6 = b_7 = \cdots = 0 \end{aligned} \tag{3.15}$$

is a Casimir operator of the algebra g_{d} defined in (3.1).

3.3. Unitary irreducible representations

In this subsection we will study the deformations of some unitary irreducible representations (unirreps) of the Lie algebras $g = \mathrm{so}(4)$, so(3,1) and e(3) when the algebras are replaced by their corresponding deformed algebras g_{d}.

The unirreps considered are those which have a representation space $\mathcal{R}$ that contains each of the representation spaces $\mathcal{R}^l$, $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, of so(3) at most once (Biedenharn 1961, Naimark 1964, Böhm 1979). Such unirreps can be characterized by

(i) $[p, q]$ where $p \geqslant |q|$, and $p, |q| \in \mathbb{N}$ or $p, |q| \in \frac{1}{2}\mathbb{N}$, in the so(4) case;

(ii) (l_0, c) where either $l_0 \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}$ and $c \in \mathbb{R}$, or $l_0 = 0$ and $c = \mathrm{i}\nu$, $\nu \in \mathbb{R}$, in the so(3,1) case;

(iii) (l_0, ϵ) where $l_0 \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}$ and $\epsilon \in \mathbb{R}$, in the e(3) case, where the latter are obtained from those of so(3,1) by an Inönü–Wigner contraction.

In cases (i) and (ii) (or (iii)) the decomposition of their representation space is given by

$$\mathcal{R} = \sum_{l=l_0,l_0+1}^{l_1} \oplus \mathcal{R}^l \tag{3.16a}$$

$$\mathcal{R} = \sum_{l=l_0,l_0+1}^{\infty} \oplus \mathcal{R}^l \tag{3.16b}$$

where in (3.16a) the minimum l value is defined by $l_0 = |q|$ and the maximum l value is defined by $l_1 = p$.

The reduced matrix elements of the vector operator A and the eigenvalues of the Casimir operators can be written as†

$$\begin{aligned} \langle [p,q]l \| A \| [p,q]l \rangle &= \frac{q(p+1)}{[l(l+1)]^{1/2}} \\ \langle [p,q]l-1 \| A \| [p,q]l \rangle &= -\left[\frac{(l-q)(l+q)(p+1-l)(p+1+l)}{l(2l-1)} \right]^{1/2} \end{aligned} \tag{3.17}$$

and

$$\langle C_1 \rangle = p(p+2) + q^2 \qquad \langle C_2 \rangle = q(p+1) \tag{3.18}$$

for so(4),

$$\begin{aligned} \langle (l_0,c)l \| A \| (l_0,c)l \rangle &= -\frac{l_0 c}{[l(l+1)]^{1/2}} \\ \langle (l_0,c)l-1 \| A \| (l_0,c)l \rangle &= -\left[\frac{(l-l_0)(l+l_0)(c^2+l^2)}{l(2l-1)} \right]^{1/2} \end{aligned} \tag{3.19}$$

and

$$\langle C_1 \rangle = c^2 - l_0^2 + 1 \qquad \langle C_2 \rangle = -l_0 c \tag{3.20}$$

for so(3,1),

$$\begin{aligned} \langle (l_0,\epsilon)l \| A \| (l_0,\epsilon)l \rangle &= -\frac{l_0 \epsilon}{[l(l+1)]^{1/2}} \\ \langle (l_0,\epsilon)l-1 \| A \| (l_0,\epsilon)l \rangle &= -|\epsilon| \left[\frac{(l-l_0)(l+l_0)}{l(2l-1)} \right]^{1/2} \end{aligned} \tag{3.21}$$

and

$$\langle C_1 \rangle = \epsilon^2 \qquad \langle C_2 \rangle = -l_0 \epsilon \tag{3.22}$$

for e(3). In all cases the remaining reduced matrix elements of A can be obtained from the relation

$$\langle l+1 \| A \| l \rangle = -\left(\frac{2l+1}{2l+3} \right)^{1/2} \langle l \| A \| l+1 \rangle \tag{3.23}$$

valid for any vector operator.

By introducing the function $G(l^2, l_0^2)$, defined by

$$\begin{aligned} G(l^2, l_0^2) &= \frac{1}{l^2 - l_0^2} \sum_{j=l_0}^{l-1} (2j+1) g\,(j(j+1)) \\ &= a_0 + \tfrac{1}{2} a_1 (l^2 + l_0^2 - 1) + \tfrac{1}{3} a_2 \left[(l^2-1)^2 + l_0^2(l^2-1) + l_0^2(l_0^2-1) \right] + \cdots \end{aligned} \tag{3.24}$$

for $l > l_0$, we easily find the following results.

† The phase convention adopted in this paper is that of Biedenharn (1961) which differs from that of Naimark (1964) and Böhm (1979).

Proposition 4. Provided that the deformation parameters a_k $k > 0$, are chosen in such a way that all quantities under square roots remain non-negative, then the unirreps $[p, q]$, (l_0, c) and (l_0, ϵ) of so(4), so(3,1) and e(3) respectively, can be deformed into unirreps of the corresponding deformed algebras g_d. The reduced matrix elements of A and the eigenvalues of the Casimir operators become

$$\begin{aligned}
\langle [p,q]l \| A \| [p,q]l \rangle &= q(p+1) \left[\frac{G\left((p+1)^2, q^2\right)}{l(l+1)} \right]^{1/2} \\
\langle [p,q]l-1 \| A \| [p,q]l \rangle &= -\left\{ \frac{(l-q)(l+q)\left[(p+1)^2 G\left((p+1)^2, q^2\right) - l^2 G\left(l^2, q^2\right)\right]}{l(2l-1)} \right\}^{1/2}
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
\langle C_{1d} \rangle &= h\left(|q|(|q|+1)\right) - (|q|+1) G\left((|q|+1)^2, q^2\right) + (p+1)^2 G\left((p+1)^2, q^2\right) \\
&= \langle C_1 \rangle + \tfrac{1}{2} a_1 \left[\langle C_1 \rangle \left(\langle C_1 \rangle + 1\right) - \langle C_2 \rangle^2\right] + \tfrac{1}{3} a_2 \langle C_1 \rangle \left[\langle C_1 \rangle \left(\langle C_1 \rangle + 1\right) - 2\langle C_2 \rangle^2\right] \\
&\quad + \cdots \\
\langle C_{2d} \rangle &= q(p+1) \left[G\left((p+1)^2, q^2\right)\right]^{1/2} \\
&= \langle C_2 \rangle \left[1 + \tfrac{1}{2} a_1 \langle C_1 \rangle + \tfrac{1}{3} a_2 \left(\langle C_1 \rangle^2 - \langle C_2 \rangle^2\right) + \cdots\right]^{1/2}
\end{aligned} \tag{3.26}$$

in the deformed so(4) case,

$$\begin{aligned}
\langle (l_0, c)l \| A \| (l_0, c)l \rangle &= -\frac{l_0 c}{[l(l+1)]^{1/2}} \\
\langle (l_0, c)l-1 \| A \| (l_0, c)l \rangle &= -\left\{ \frac{(l-l_0)(l+l_0)\left[c^2 - l^2 G(l^2, l_0^2)\right]}{l(2l-1)} \right\}^{1/2}
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
\langle C_{1d} \rangle &= h\left(l_0(l_0+1)\right) - (l_0+1) G\left((l_0+1)^2, l_0^2\right) + c^2 \\
&= \langle C_1 \rangle + \tfrac{1}{2} a_1 l_0^2 \left(l_0^2 - 1\right) + \tfrac{1}{3} a_2 l_0^2 \left(l_0^2 - 1\right)^2 + \cdots \\
\langle C_{2d} \rangle &= \langle C_2 \rangle = -l_0 c
\end{aligned} \tag{3.28}$$

in the deformed so(3,1) case, while for deformed e(3) they can be obtained by substituting ϵ for c in equations (3.27) and (3.28).

Proof. Whenever all a_k's for which $k > 0$ go to zero, equations (3.25)–(3.28) and their counterparts for e(3) transform into the undeformed results contained in equations (3.17)–(3.22) respectively. Alternatively, for arbitrary values of the a_k's satisfying the hypothesis the validity of the equations can be checked by direct substitution into the commutation relation (3.1c) and the definitions of C_{1d} and C_{2d}. □

Remark. For some choices of the deformation parameters it may happen that the unirreps of g_d considered in proposition 4 do not exhaust the class of unirreps which have a representation space that contains each of the representation spaces of so(3) at most once. The existence of 'extra' representations, which have no counterpart for the undeformed algebra, has already been noted in the deformed su(2) case (Roček 1991).

4. The quadrupole case

In the quadrupole case the deformed algebra g_d is defined by the commutation relations

$$[L, L]^1 = -\sqrt{2}L \tag{4.1a}$$

$$[L, Q]^\Lambda = -\sqrt{6}\,\delta_{\Lambda,2}Q \tag{4.1b}$$

$$[Q, Q]^1 = f^1(L) = 3\sqrt{10}\,g_1(L^2)L = 3\sqrt{10}\left(\sum_{k=0}^{\infty} a_k^{(1)} L^{2k}\right)L \tag{4.1c}$$

$$\begin{aligned}[Q, Q]^3 = f^3(L) &= g_3(L^2)\left[[L \times L]^2 \times L\right]^3 \\ &= \left(\sum_{k=0}^{\infty} a_k^{(3)} L^{2k}\right)\left[[L \times L]^2 \times L\right]^3\end{aligned} \tag{4.1d}$$

where the normalization constant γ_1 has been set equal to $3\sqrt{10}$. The algebra is associative providing that Q satisfies equations (2.7*a*) and (2.7*b*).

In the undeformed case where the algebra g_d reduces to an ordinary Lie algebra g, one has $a_k^{(1)} = 0, k = 1, 2, \ldots$, and $a_k^{(3)} = 0$, $k = 0, 1, \ldots$. According to whether $a_0^{(1)} = +1$, -1, or 0, g is the special unitary algebra su(3) (Elliott 1958a,b), the special linear algebra sl(3,R) (Weaver and Biedenharn 1972) or the semidirect sum of an Abelian algebra t(5) and so(3) (Ui 1970, Weaver *et al* 1973).

If we restrict ourselves to a first-order deformation, equations (4.1*c*) and (4.1*d*) become

$$[Q, Q]^1 = 3\sqrt{10}\,\epsilon L + \alpha L^2 L \tag{4.2a}$$

$$[Q, Q]^3 = \beta\left[[L \times L]^2 \times L\right]^3 \tag{4.2b}$$

where α, β and ϵ are defined by $\alpha = 3\sqrt{10}a_1^{(1)}$, $\beta = a_0^{(3)}$ and $\epsilon = a_0^{(1)} = +1, -1$ or 0. By substituting equations (4.2*a*) and (4.2*b*) into equations (2.7a) and (2.7*b*) and taking equation (4.1b) into account we obtain the two associativity conditions

$$2\sqrt{2}\alpha\left[Q, L^2L\right]^1 + \sqrt{7}\beta\left[Q, \left[[L \times L]^2 \times L\right]^3\right]^1 = 0 \tag{4.3a}$$

$$\alpha\left[Q, L^2L\right]^3 - 2\beta\left[Q, \left[[L \times L]^2 \times L\right]^3\right]^3 = 0. \tag{4.3b}$$

Straightforward tensor algebra leads to the following results:

$$\begin{aligned}\left[Q, L^2L\right]^\Lambda &= 2\sqrt{3}\left\{\left[\sqrt{3} - U(11\Lambda2; 12)\right][L \times Q]^\Lambda \right.\\ &\quad \left. + \sqrt{2}\,U(11\Lambda2; 22)\left[[L \times L]^2 \times Q\right]^\Lambda\right\} \\ \left[Q, \left[[L \times L]^2 \times L\right]^3\right]^\Lambda &= \sqrt{3}\left\{\left[\sqrt{7}\,U(22\Lambda1; 23)\right.\right.\\ &\quad \left. - 2\sqrt{2\Lambda(\Lambda+1)}\,U(22\Lambda1; \Lambda3)U(21\Lambda1; 22)\right][L \times Q]^\Lambda \\ &\quad \left. + 3\sqrt{2}\,U(21\Lambda2; 23)\left[[L \times L]^2 \times Q\right]^\Lambda\right\}\end{aligned} \tag{4.4}$$

valid for $\Lambda = 1$ or 3. By replacing Racah coefficients by their numerical values in equation (4.4), conditions (4.3*a*) and (4.3*b*) can be rewritten as

$$\left(6\sqrt{10}\alpha - 7\beta\right)[L \times Q]^1 + \sqrt{6}\left(2\sqrt{10}\alpha + 7\beta\right)\left[[L \times L]^2 \times Q\right]^1 = 0 \tag{4.5a}$$

$$2\left(\sqrt{10}\alpha + 3\beta\right)[L \times Q]^3 + \left(\sqrt{10}\alpha - 9\beta\right)\left[[L \times L]^2 \times Q\right]^3 = 0. \tag{4.5b}$$

Since they cannot be satisfied for any choice of the deformation parameters α and β we conclude that for a first-order deformation the algebra g_d, defined in (4.1), is not associative contrary to what occurs for the algebra (3.1) in the vector case. Therefore, we shall not pursue the analysis of the quadrupole case any further.

5. Conclusion

In this paper we have established that deformations of the Lie algebras so(4), so(3,1) and e(3) that leave their so(3) subalgebra undeformed and preserve their coset structure exist. We have proved that the Casimir operators of these Lie algebras can be deformed so as to provide the corresponding operators for the deformed algebras. Moreover, we have constructed those unirreps of the deformed algebras that, for vanishing deformation, transform into the unirreps of the corresponding Lie algebras belonging to an important class of representations.

In contrast, we have shown that a similar deformation of the Lie algebras su(3), sl(3,$\mathbb{R}$) and of the semidirect sum of t(5) and so(3) is not possible because the associativity conditions are violated for a first-order deformation.

It should be stressed that the deformations of so(4), so(3,1), and e(3) studied here differ from the standard q-algebras so$_q$(4), so$_q$(3,1) and e$_q$(3) (Drinfeld 1986, Jimbo 1985, Celeghini *et al* 1991, Chakrabarti 1993), as well as from an alternative deformation of the orthogonal and pseudo-orthogonal Lie algebras proposed by Gavrilik and Klimyk (1991) (see also Gavrilik 1993). In both these approaches, the so(3) subalgebra is indeed deformed contrary to what occurs in the algebras considered here. Since rotational invariance and angular-momentum conservation are important properties of many physical systems, one may hope that the deformed algebras introduced in this paper will prove more relevant to applications than those previously considered.

Some problems where the coset structure so(4) and/or so(3) is important can be found in standard quantum mechanics (e.g. that of a particle in a Coulomb potential (Biedenharn 1961)), as well as in parasupersymmetric quantum mechanics with three parasupercharges (Debergh and Nikitin 1995). Deformations of so(4) preserving that coset structure may therefore be expected to play a role in similar contexts. It is already known that the first-order deformation of so(4) is the symmetry algebra of a particle in a Coulomb potential when the space has a constant curvature (Higgs 1979, Leemon 1979, Granovskii *et al* 1992, de Vos and van Driel 1993). All of the general results derived in this paper, therefore, apply to such a problem. At a more phenomenological level, deviations from hydrogenic spectra that are found for many-electron atoms or excitons in semiconductors might be accounted for by some deformations of so(4). Similarly, deformations of parasupersymmetric quantum mechanics with three parasupercharges might lead to some parasupersymmetric Hamiltonians with new and non-trivial properties as happens in the case of two parasupercharges (Debergh and Nikitin 1995). We hope to return to some of these problems in forthcoming publications.

Appendix 1. Definition and properties of coupled commutators

The purpose of this appendix is to review the definition and some useful properties of coupled commutators.

The coupled commutator of two so(3) irreducible tensors T^{λ_1} and U^{λ_2}, of ranks λ_1 and λ_2 respectively, is defined by

$$\left[T^{\lambda_1}, U^{\lambda_2}\right]^{\Lambda}_{M} = \sum_{\mu_1\mu_2} \langle \lambda_1\mu_1, \lambda_2\mu_2|\Lambda M\rangle \left[T^{\lambda_1}_{\mu_1}, U^{\lambda_2}_{\mu_2}\right] \tag{A1.1}$$

in terms of an ordinary commutator $[\,,\,]$ and of an su(2) Wigner coefficient $\langle\,,\,|\,\rangle$. By using a symmetry property of the Wigner coefficient (Rose 1957) equation (A1.1) can be written alternatively as

$$\begin{aligned}\left[T^{\lambda_1}, U^{\lambda_2}\right]^{\Lambda}_{M} &= \left[T^{\lambda_1} \times U^{\lambda_2}\right]^{\Lambda}_{M} - (-1)^{\lambda_1+\lambda_2-\Lambda}\left[U^{\lambda_2} \times T^{\lambda_1}\right]^{\Lambda}_{M} \\ &= -(-1)^{\lambda_1+\lambda_2-\Lambda}\left[U^{\lambda_2}, T^{\lambda_1}\right]^{\Lambda}_{M}\end{aligned} \tag{A1.2}$$

where

$$\left[T^{\lambda_1} \times U^{\lambda_2}\right]^{\Lambda}_{M} = \sum_{\mu_1\mu_2} \langle \lambda_1\mu_1, \lambda_2\mu_2|\Lambda M\rangle\, T^{\lambda_1}_{\mu_1} U^{\lambda_2}_{\mu_2}. \tag{A1.3}$$

For three irreducible tensors T^{λ_1}, U^{λ_2} and V^{λ_3} the well known relation $[A, BC] = [A, B]C + B[A, C]$ becomes in coupled form

$$\begin{aligned}\left[T^{\lambda_1}, \left[U^{\lambda_2} \times V^{\lambda_3}\right]^{\Lambda_{23}}\right]^{\Lambda}_{M} &= \sum_{\Lambda_{12}} U(\lambda_1\lambda_2\Lambda\lambda_3; \Lambda_{12}\Lambda_{23}) \left[\left[T^{\lambda_1}, U^{\lambda_2}\right]^{\Lambda_{12}} \times V^{\lambda_3}\right]^{\Lambda}_{M} \\ &\quad + \sum_{\Lambda_{13}} (-1)^{\lambda_3+\Lambda-\Lambda_{13}-\Lambda_{23}} U(\lambda_1\lambda_3\Lambda\lambda_2; \Lambda_{13}\Lambda_{23}) \left[U^{\lambda_2} \times \left[T^{\lambda_1}, V^{\lambda_3}\right]^{\Lambda_{13}}\right]^{\Lambda}_{M}\end{aligned} \tag{A1.4}$$

where $U(\lambda_1\lambda_2\Lambda\lambda_3; \Lambda_{12}\Lambda_{23})$ denotes a Racah coefficient in unitary form (Rose 1957). In the same way, the Jacobi identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ can be rewritten as

$$\begin{aligned}\left[T^{\lambda_1}, \left[U^{\lambda_2}, V^{\lambda_3}\right]^{\Lambda_{23}}\right]^{\Lambda}_{M} &+ \sum_{\Lambda_{31}} (-1)^{\lambda_1+\Lambda_{23}-\Lambda} U(\lambda_2\lambda_3\Lambda\lambda_1; \Lambda_{23}\Lambda_{31}) \left[U^{\lambda_2}, \left[V^{\lambda_3}, T^{\lambda_1}\right]^{\Lambda_{31}}\right]^{\Lambda}_{M} \\ &+ \sum_{\Lambda_{12}} (-1)^{\lambda_3+\Lambda_{12}-\Lambda} U(\lambda_1\lambda_2\Lambda\lambda_3; \Lambda_{12}\Lambda_{23}) \left[V^{\lambda_3}, \left[T^{\lambda_1}, U^{\lambda_2}\right]^{\Lambda_{12}}\right]^{\Lambda}_{M} = 0.\end{aligned} \tag{A1.5}$$

Appendix 2. Proofs of lemmas 1 and 2 and of proposition 3

To prove equations (3.11) and (3.12), we proceed by induction over k. For the lowest value $k = 1$, equation (3.11) reduces to (3.4). For $k > 1$ we start from the identities

$$\left[A, L^{2k}\right]^1 = L^{2k-2}\left[A, L^2\right]^1 + L^2\left[A, L^{2k-2}\right]^1 + \left[\left[A, L^{2k-2}\right]^1, L^2\right]^1 \tag{A2.1}$$

and

$$\left[\left[A, L^{2k-2}\right]^1, L^2\right]^1 = 2\left\{\left[A, L^{2k-2}\right]^1 + \sqrt{2}\left[L \times \left[A, L^{2k-2}\right]^1\right]^1\right\} \tag{A2.2}$$

resulting from equation (A1.4) and standard tensor algebra. Then, assuming equation (3.11) to be valid when k is replaced by $k-1$, we use it to compute the right-hand sides of equations (A2.1) and (A2.2). The result contains two tensor products that are not in the same standard form as those appearing on the right-hand side of (3.11) but they can be rewritten using the following identities:

$$\left[L \times [L \times A]^1\right]^1 = \frac{1}{3}L^2 A - \frac{1}{2\sqrt{2}}[L \times A]^1 + \frac{1}{2}\sqrt{\frac{5}{3}}\left[[L \times L]^2 \times A\right]^1 \tag{A2.3}$$

$$\left[L \times \left[[L \times L]^2 \times A\right]^1\right]^1 = -\frac{1}{4\sqrt{15}}\left(3 - 4L^2\right)[L \times A]^1 - \frac{3}{2\sqrt{2}}\left[[L \times L]^2 \times A\right]^1. \tag{A2.4}$$

After some straightforward calculations equation (3.11) is obtained, provided that $x_i^{(k)}$, $y_i^{(k)}$, and $z_i^{(k)}$ satisfy equation (3.12).

The proofs of equations (3.13) and (3.14) are based upon the identities

$$\left[A, A^2\right] = -2\sqrt{2}\sum_{k=0}^{\infty} a_k L^{2k}[L \times A]^1 + \sqrt{2}\sum_{k=0}^{\infty} a_k \left[A, L^{2k}L\right]^1 \tag{A2.5}$$

and

$$\left[A, L^{2k}L\right]^l = -\sqrt{2}\left[A, L^{2k}\right]^l - \left[L \times \left[A, L^{2k}\right]^1\right]^1 - \sqrt{2}L^{2k}A \tag{A2.6}$$

resulting from equations (A1.4) and (3.1), as well as standard tensor algebra. Taking equations (3.11), (A2.3) and (A2.4) into account leads directly to the required results.

Finally, from equations (3.10), (3.11) and (3.13) it follows that for arbitrary constants a_k the condition $[A, C_{1d}]^1 = 0$ is fulfilled provided that

$$\sum_{k=i+1}^{\infty} b_k x_i^{(k)} - u_i = \sum_{k=i+1}^{\infty} b_k y_i^{(k)} - v_i = \sum_{k=i+2}^{\infty} b_k z_i^{(k)} - w_i = 0 \qquad i = 0, 1, 2, \ldots. \tag{A2.7}$$

By successive use of equations (3.14) and (3.12) these conditions can be rewritten as

$$2\sum_{k=i+1}^{\infty} b_k' x_i^{(k)} = \sum_{k=i}^{\infty} a_k\left(2x_i^{(k)} - x_{i-1}^{(k)} + 2\delta_{k,i}\right) \qquad i = 0, 1, 2, \ldots \tag{A2.8a}$$

$$2\sum_{k=i+1}^{\infty} b_k' y_i^{(k)} = \sum_{k=i}^{\infty} a_k\left(2y_i^{(k)} - y_{i-1}^{(k)} + 2\sqrt{2}\delta_{k,i}\right) \qquad i = 0, 1, 2, \ldots \tag{A2.8b}$$

$$2\sum_{k=i+2}^{\infty} b_k' z_i^{(k)} = \sum_{k=i+1}^{\infty} a_k\left(2z_i^{(k)} - z_{i-1}^{(k)}\right) \qquad i = 0, 1, 2, \ldots \tag{A2.8c}$$

where $b_k' \equiv b_k - \frac{1}{2}a_{k-1}$.

If we restrict ourselves to a Kth-order deformation and assume that $b_k = 0$ for $k > K+1$, conditions (A2.8a) and (A2.8b) reduce to two systems of $K+1$ equations (corresponding to $i = 0, 1, \ldots, K$) in $K+1$ unknowns b_k', $k = 1, 2, \ldots, K+1$, while condition (A2.8c) leads to a system of K equations (corresponding to $i = 0, 1, \ldots, K-1$) in K unknowns b_k', $k = 2, 3, \ldots, K+1$. It only remains to solve (A2.8a) and to check that its solution also satisfies the two remaining systems of equations. This calculation was carried out for $K = 4$ and the results are contained in equation (3.15).

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